A NEW CHARACTERIZATION OF q_ω-COMPACT ALGEBRAS

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Abstract. In this note, we give a new characterization for an algebra to be q_ω-compact in terms of *super-product operations* on the lattice of congruences of the relative free algebra.

AMS Subject Classification Primary 03C99, Secondary 08A99 and 14A99.
Keywords algebraic structures; equations; algebraic set; radical ideal; q_ω-compactness; filter-power; geometric equivalence; relatively free algebra; quasi-identity; quasi-variety.

1. Introduction

In this article, our notations are the same as [2], [3], [4], [5] and [6]. The reader should review these references for a complete account of the universal algebraic geometry. However, a brief review of fundamental notions will be given in the next section.

Let $\mathcal{L}$ be an algebraic language, A be an algebra of type $\mathcal{L}$ and S be a system of equation in the language $\mathcal{L}$. Recall that an equation $p \approx q$ is a logical consequence of S with respect to A, if any solution of S in A is also a solution of $p \approx q$. The radical $\mathrm{Rad}_A(S)$ is the set of all logical consequences of S with respect to A. This radical is clearly a congruence of the term algebra $T_{\mathcal{L}}(X)$ and in fact it is the largest subset of the term algebra which is equivalent to S with respect to A. Generally, this logical system of equations with respect to A does not obey the ordinary compactness of the first order logic. We say that an algebra A is q_ω-compact, if for any system S and any consequence $p \approx q$, there exists a finite subset $S_0 \subseteq S$ with the property that $p \approx q$ is a consequence of S_0 with respect to A. This property of being q_ω-compact is equivalent to

$$\mathrm{Rad}_A(S) = \bigcup_{S_0} \mathrm{Rad}_A(S_0),$$

where S_0 varies in the set of all finite subsets of S. If we look at the map Rad_A as a closure operator on the lattice of systems of equations in the language $\mathcal{L}$, then we see that A is q_ω-compact if and only if Rad_A

is an algebraic. The class of q_ω-compact algebras is very important and it contains many elements. For example, all equationally noetherian algebras belong to this class. In [4], some equivalent conditions for q_ω-compactness are given. Another equivalent condition is obtained in [7] in terms of *geometric equivalence*. It is proved that (the proof is implicit in [7]) an algebra A is q_ω-compact if and only if A is geometrically equivalent to any of its filter-powers. We will discuss geometric equivalence in the next section. We will use this fact of [7] to obtain a new characterization of q_ω-compact algebras. Although our main result will be formulated in an arbitrary variety of algebras, in this introduction, we give a simple description of this result for the case of the variety of all algebras of type $\mathcal{L}$.

Roughly speaking, a *super-product operation* is a map C which takes a set K of congruences of the term algebra and returns a new congruence $C(K)$ such that for all $\theta \in K$, we have $\theta \subseteq C(K)$. For an algebra B define a map T_B which takes a system S of equations and returns

$$T_B(S) = \{\mathrm{Rad}_B(S_0) : \ S_0 \subseteq S, \ |S_0| < \infty\}.$$

Suppose for all algebra B we have $C \circ T_B \leq \mathrm{Rad}_B$. We prove that an algebra A is q_ω-compact if and only if $C \circ T_A = \mathrm{Rad}_A$.

2. Main result

Suppose $\mathcal{L}$ is an algebraic language. All algebras we are dealing with, are of type $\mathcal{L}$. Let $\mathbf{V}$ be a variety of algebras. For any $n \geq 1$, we denote the relative free algebra of $\mathbf{V}$, generated by the finite set $X = \{x_1, \ldots, x_n\}$, by $F_{\mathbf{V}}(n)$. Clearly, we can assume that an arbitrary element $(p, q) \in F_{\mathbf{V}}(n)^2$ is an equation in the variety $\mathbf{V}$ and we can denote it by $p \approx q$. We introduce the following list of notations:

1- $\mathrm{P}(F_{\mathbf{V}}(n)^2)$ is the set of all systems of equations in the variety $\mathbf{V}$.

2- $\mathrm{Con}(F_{\mathbf{V}}(n))$ is the set of all congruences of $F_{\mathbf{V}}(n)$.

3- $\Sigma(\mathbf{V}) = \bigcup_{n=1}^{\infty} P(F_{\mathbf{V}}(n)^2)$.

4- $\mathrm{Con}(\mathbf{V}) = \bigcup_{n=1}^{\infty} \mathrm{Con}(F_{\mathbf{V}}(n))$.

5- $\mathrm{PCon}(\mathbf{V}) = \bigcup_{n=1}^{\infty} \mathrm{P}(\mathrm{Con}(F_{\mathbf{V}}(n)))$.

6- $q_\omega(\mathbf{V})$ is the set of all q_ω-compact elements of $\mathbf{V}$.

Note that, we have $\mathrm{Con}(\mathbf{V}) \subseteq \Sigma(\mathbf{V})$. For any algebra $B \in \mathbf{V}$, the

map $\mathrm{Rad}_B : \Sigma(\mathbf{V}) \to \Sigma(\mathbf{V})$ is a closure operator and B is q_ω-compact, if and only if this operator is algebraic. Define a map

$$T_B : \Sigma(\mathbf{V}) \to \mathrm{PCon}(\mathbf{V})$$

by

$$T_B(S) = \{\mathrm{Rad}_B(S_0) : \ S_0 \subseteq S, \ |S_0| < \infty\}.$$

Definition 1. *A map C : $\mathrm{PCon}(\mathbf{V}) \to \mathrm{Con}(\mathbf{V})$ is called a super-product operation, if for any $K \in \mathrm{PCon}(\mathbf{V})$ and $\theta \in K$, we have $\theta \subseteq C(K)$.*

There are many examples of such operations; the ordinary product of normal subgroups in the varieties of groups is the simplest one. For another example, we can look at the map $C(K) = \mathrm{Rad}_B(\bigcup_{\theta\in K} \theta)$, for a given fixed $B \in \mathbf{V}$. We are now ready to present our main result.

Theorem 1. *Let C be a super-product operation such that for any $B \in \mathbf{V}$, we have $C \circ T_B \leq \mathrm{Rad}_B$. Then*

$$q_\omega(\mathbf{V}) = \{A \in \mathbf{V} : \ C \circ T_A = \mathrm{Rad}_A\}.$$

To prove the theorem, we first give a proof for the following claim. Note that it is implicitly proved in [7] for the case of groups.

An algebra is q_ω-compact if and only if it is geometrically equivalent to any of its filter-powers.

Let $A \in \mathbf{V}$ be a q_ω-compact algebra and I be a set of indices. Let $F \subseteq P(I)$ be a filter and $B = A^I/F$ be the corresponding filter-power. We know that the quasi-varieties generated by A and B are the same. So, these algebras have the same sets of quasi-identities. Now, suppose that S_0 is a finite system of equations and $p \approx q$ is another equation. Consider the following quasi-identity

$$\forall \overline{x}(S_0(\overline{x}) \to p(\overline{x}) \approx q(\overline{x})).$$

This quasi-identity is true in A, if and only if it is true in B. This shows that $\mathrm{Rad}_A(S_0) = \mathrm{Rad}_B(S_0)$. Now, for an arbitrary system S, we have

$$\begin{aligned}\mathrm{Rad}_A(S) &= \bigcup_{S_0} \mathrm{Rad}_A(S_0)\\ &= \bigcup_{S_0} \mathrm{Rad}_B(S_0)\\ &\subseteq \mathrm{Rad}_B(S).\end{aligned}$$

Note that in the above equalities, S_0 ranges in the set of finite subsets of S. Clearly, we have $\mathrm{Rad}_B(S) \subseteq \mathrm{Rad}_A(S)$, since $A \leq B$. This shows that A and B are geometrically equivalent. To prove the converse, we need to define some notions. Let $\mathfrak{X}$ be a prevariety, i.e. a class of algebras closed under product and subalgebra. For any $n \geq 1$, let $F_{\mathfrak{X}}(n)$ be the free element of $\mathfrak{X}$ generated by n elements. Note that if $\mathbf{V} = var(\mathfrak{X})$, then $F_{\mathfrak{X}}(n) = F_{\mathbf{V}}(n)$. A congruence R in $F_{\mathfrak{X}}(n)$ is called an $\mathfrak{X}$-radical, if $F_{\mathfrak{X}}(n)/R \in \mathfrak{X}$. For any $S \subseteq F_{\mathfrak{X}}(n)^2$, the least $\mathfrak{X}$-radical containing S is denoted by $\mathrm{Rad}_{\mathfrak{X}}(S)$.

Lemma 1. *For an algebra A and any system S, we have*

$$\mathrm{Rad}_A(S) = \mathrm{Rad}_{pvar(A)}(S),$$

where $pvar(A)$ is the prevariety generated by A.

Proof. Since $F_{\mathfrak{X}}(n)/\mathrm{Rad}_A(S)$ is a coordinate algebra over A, so it embeds in a direct power of A and hence it is an element of $pvar(A)$. This shows that

$$\mathrm{Rad}_{pvar(A)}(S) \subseteq \mathrm{Rad}_A(S).$$

Now, suppose (p, q) does not belong to $\mathrm{Rad}_{pvar(A)}(S)$. So, there exists $B \in pvar(A)$ and a homomorphism $\varphi : F_{\mathfrak{X}}(n) \to B$ such that $S \subseteq \ker \varphi$ and $\varphi(p) \neq \varphi(q)$. But, B is separated by A, hence there is a homomorphism $\psi : B \to A$ such that $\psi(\varphi(p)) \neq \psi(\varphi(q))$. This shows that (p, q) does not belong to $\ker(\psi \circ \varphi)$. Therefore, it is not in $\mathrm{Rad}_A(S)$. $\square$

Note that, since $pvar(A)$ is not axiomatizable in general, so we can not give a deductive description of elements of $\mathrm{Rad}_A(S)$. But, for $\mathrm{Rad}_{var(A)}(S)$ and $\mathrm{Rad}_{qvar(A)}(S)$ this is possible, because the variety and quasi-variety generated by A are axiomatizable. More precisely, we have:

1- Let $\mathrm{Id}(A)$ be the set of all identities of A. Then $\mathrm{Rad}_{var(A)}(S)$ is the set of all logical consequences of S and $\mathrm{Id}(A)$.

2- Let $\mathrm{Q}(A)$ be the set of all identities of A. Then $\mathrm{Rad}_{qvar(A)}(S)$ is the set of all logical consequences of S and $\mathrm{Q}(A)$.

We can now, prove the converse of the claim. Suppose A is not q_ω-compact. We show that

$$pvar(A)_\omega \neq qvar(A)_\omega.$$

Recall that for and arbitrary class $\mathfrak{X}$, the notation $\mathfrak{X}_\omega$ denotes the class of finitely generated elements of $\mathfrak{X}$. Suppose in contrary we have the

equality
$$pvar(A)_\omega = qvar(A)_\omega.$$
Assume that S is an arbitrary system and $(p, q) \in \mathrm{Rad}_A(S)$. Hence, the infinite quasi-identity
$$\forall \overline{x}(S(\overline{x}) \to p(\overline{x}) \approx q(\overline{x}))$$
is true in A. So, it is also true in $pvar(A)$. As a result, every element from $qvar(A)_\omega$ satisfies this infinite quasi-identity. Let $F_A(n) = F_{var(A)}(n)$. We have $F_A(n) \in qvar(A)_\omega$ and hence $\mathrm{Rad}_{qvar(A)}(S)$ depends only on $qvar(A)_\omega$. In other words, $(p, q) \in \mathrm{Rad}_{qvar(A)}(S)$, so $p \approx q$ is a logical consequence of the set of $S + \mathrm{Q}(A)$. By the compactness theorem of the first order logic, there exists a finite subset $S_0 \subseteq S$ such that $p \approx q$ is a logical consequence of $S_0 + \mathrm{Q}(A)$. This shows that $(p, q) \in \mathrm{Rad}_{qvar(A)}(S_0)$. But $\mathrm{Rad}_{qvar(A)}(S_0) \subseteq \mathrm{Rad}_A(S_0)$. Hence $(p, q) \in \mathrm{Rad}_A(S_0)$, violating our assumption of non-q_ω-compactness of A. We now showed that
$$pvar(A)_\omega \neq qvar(A)_\omega.$$
By the algebraic characterizations of the classes $pvar(A)$ and $qvar(A)$, we have
$$SP(A)_\omega \neq SPP_u(A)_\omega,$$
where P_u is the ultra-product operation. This shows that there is an ultra-power B of A such that
$$SP(A)_\omega \neq SP(B)_\omega.$$
In other words the classes $pvar(A)_\omega$ and $pvar(B)_\omega$ are different. We claim that A and B are not geometrically equivalent. Suppose this is not the case. Let $A_1 \in pvar(A)_\omega$. Then A_1 is a coordinate algebra over A, i.e. there is a system S such that
$$A_1 = \frac{F_{\mathbf{V}}(n)}{\mathrm{Rad}_A(S)}.$$
Since $\mathrm{Rad}_A(S) = \mathrm{Rad}_B(S)$, so
$$A_1 = \frac{F_{\mathbf{V}}(n)}{\mathrm{Rad}_B(S)},$$
and hence A_1 is a coordinate algebra over B. This argument shows that
$$pvar(A)_\omega = pvar(B)_\omega,$$
which is a contradiction. Therefore A and B are not geometrically equivalent and this completes the proof of the claim. We can now complete the proof of the theorem. Assume that $C \circ T_A = \mathrm{Rad}_A$. We show that A is geometrically equivalent to any of its filter-powers. So,

let $B = A^I/F$ be a filter-power of A. Note that we already proved that for a finite system S_0, the radicals $\mathrm{Rad}_A(S_0)$ and $\mathrm{Rad}_B(S_0)$ are the same. Suppose that S is an arbitrary system of equations. We have

$$\begin{aligned}\mathrm{Rad}_A(S) &= C(T_A(S))\\ &= C(\{\mathrm{Rad}_A(S_0):\ S_0 \subseteq S, |S_0| < \infty\})\\ &= C(\{\mathrm{Rad}_B(S_0):\ S_0 \subseteq S, |S_0| < \infty\})\\ &\subseteq \mathrm{Rad}_B(S).\end{aligned}$$

So we have $\mathrm{Rad}_A(S) = \mathrm{Rad}_B(S)$ and hence A and B are geometrically equivalent. This shows that A is q_ω-compact. Conversely, let A be q_ω-compact. For any system S, we have

$$\begin{aligned}\mathrm{Rad}_A(S) &= \bigcup_{S_0} \mathrm{Rad}_A(S_0)\\ &= \bigvee\{\mathrm{Rad}_A(S_0) : S_0 \subseteq S, |S_0| < \infty\}\\ &= \bigvee T_A(S),\end{aligned}$$

where $\bigvee$ denotes the least upper bound. By our assumption, $C(T_A(S)) \subseteq \mathrm{Rad}_A(S)$, so $C(T_A(S)) \subseteq \bigvee T_A(S)$. On the other hand, for any finite $S_0 \subseteq S$, we have $\mathrm{Rad}_A(S_0) \subseteq C(T_A(S))$. This shows that

$$C(T_A(S)) = \bigvee T_A(S),$$

and hence $C \circ T_A = \mathrm{Rad}_A$. The proof is now completed.

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